ON THE RELATIVE CHARACTER GRAPH OF A FINITE GROUP

Dr.K.T.NAGALAKSHMI[#1]
Department of Mathematics
K.L.N.College of Information Technology
Pottapalayam-630 611,Sivagangai District
Tamil Nadu, INDIA

Dr.A.V. JEYAKUMAR[#2]
Department of Mathematics
Periyar Maniammai University
Vallam,Tanjore District
Tamil Nadu, INDIA

***Abstract*—This paper,in a sense, is a sequel to an earlier construction by T.Gnanaseelan of a graph Γ(G, H) for any finite group G and a subgroup H using (complex) irreducible characters (See [3]). We construct another graph Ω(G, H), which is structurally quite different from Γ(G,H). However, we prove that for the special case of the sequence of subgroups $S_{n-1} \subset S_n \subset S_{n+1}$ (where S_n is the symmetric group on n letters), Γ (S_n, S_{n-1}) and Ω(S_{n+1}, S_n) are indeed isomorphic.**

Key words: Character Theory, Graph Theory, Group Theory, and Representation Theory.

I. INTRODUCTION

From the time of R.Brauer, various finite graphs have been constructed using mostly irreducible characters (both complex and p – modular) of a finite group G. These graphs in general give a pictorial representation of the intricate nature of irreducible characters of G. For instance, the famous Brauer graph has as vertex set the full set of complex irreducible characters of G and two distinct vertices are incident if and only if their reduction mod p (where p is a prime dividing O (G)) contains atleast one p-modular irreducible character in common.

Quite recently, T.Gnanaseelan in his Ph.D. thesis [17] has constructed a new finite graph for any subgroup H of G, which he calls the relative character graph of G over H and denotes by Γ (G, H). In the next section, we shall define Γ (G,H) and recount some of the salient properties as proved in [17] .

II. THE GRAPH Γ (G, H)

For the basics of character theory of G, we shall refer to [4] . The universal notations such as IrrG, χ_H, θ^G and [χ, ψ] will stand for the complete set of irreducible characters of G, the restriction of a character χ of G to a subgroup H, the induction of a character of H to G and the scalar product $\frac{1}{O(G)} \sum_{S \in G} \chi(s)\, \psi(s^{-1})$ (of course, all representations are finite dimensional taken over the complex field $\mathbb{C}$).

1) *Definition :* The relative character graph Γ (G, H) of G with respect to a subgroup H has IrrG as its vertex set and two distinct χ, ψ in IrrG are adjacent if and only if χ_H and ψ_H have atleast one element of IrrH in common. This is equivalent to saying that $[\chi, \psi]_H > 0$. Clearly Γ (G,H) is a simple graph in graph -theoretic sense, that is, it has no double edges and self-loops.

To begin with the following easy observations can be made.

2) Γ (G, H) is the null-graph if and only if H = G.

3) If H is the trivial subgroup, then Γ(G,H) is a complete graph, but even for certain types of non-trivial subgroups H, Γ (G,H) can be complete. For instance, if H is a cyclic group generated by x and if all the matrices ρ_i (x), where ρ_i runs through a full set of inequivalent irreducible representations of G, have 1 as eigen value, then Γ(G, H) turns out to be complete. (This can be of some interest in the representation theory of finite Chevalley groups).

4) If H and K are two subgroups of G such that K $\subset$ H, then Γ (G,H) is a subgraph of Γ (G,K).

5) If, $x \in G$ then Γ(G, H) = Γ (G, H^x), where H^x is the conjugate of H under x.

6) Γ(G, H) is connected if and only if core $_G$H = (1).

7) A connected graph Γ (G, H) is a tree if and only if G is a Frobenius group NH, and the kernel N is a unique elementary abelian normal p-Sylow subgroup for some prime p with order p^m and O(H) = P^m - 1.

III. THE GRAPH Ω (G, H).

We shall now construct another finite graph, again with reference to a subgroup, which, in a sense, will be dual to the graph Γ (G, H).

1) Definition : The vertex set of Ω (G, H), is IrrH and two distinct θ and ϕ are adjacent if and only if the induced characters θ^G and ϕ^G have atleast one element of IrrG in common,in other words[θ^G, ϕ^G]$_G$ > 0. The structural properties of Ω (G, H) differ in many ways to those of Γ (G, H). To begin with, even the vertices sets of both the graphs are different.

But there is one situation wherein the vertices sets of both Γ (G, H) and Ω(G, H) coincide. This occurs when we have a sequence of groups and subgroups of the form $H \subset G \subset L$. Here the vertices sets of both Γ (G, H) and Ω (L,G) coincide. But these graphs need not be the same, as can be seen from the following example. Consider the sequence H $\subset$ G $\subset$ L , where, L is the symmetric group S_4, G is the Alternating subgroup A_4 and H is the subgroup consisting of the two elements {(1), (12)(34)}. Since core $_G$H = maximal normal subgroup of G contained in H is trivial, using the criterion for connectivity obtained in [3], Γ(G,H) is a connected graph. Using Clifford's theorem, it is quite easy to see that Ω (L,G) is not connected. To see this, let 1_H, θ_2, θ_3 and θ_4 be the four distinct irreducible characters of A_4 of degrees 1, 1, 1 and 3. The conjugate action of S_4 on IrrA_4 breaks these 4 characters into 3 orbits, namely {1}, {θ_2, θ_3} and {θ_4}. Let 1_G, χ_2, χ_3, χ_4 and χ_5 be the irreducible characters of S_4 with degrees 1, 1, 2, 3 and 3 respectively.Then 1_H^G = 1_G + χ_2, θ_2^G = θ_3^G = χ_3 and θ_4^G = χ_4 + χ_5. We conclude that Ω (L,G) has three connected components, namely, {1_G, χ_2}, {χ_3} and {χ_4, χ_5}. Naturally, Γ (G,H) and Ω (G,H) are not isomorphic.

We do not propose to study the graph Ω (G,H) systematically here such as the connectivity properties etc, as was done by Gnanseelan for his graph Γ (G,H). However, taking cue from the last example, we make some beginning which partially compares with results of Gnanseelan mentioned in [7].

2) *Proposition :* If H is a normal subgroup of G, then Ω (G,H) is disconnected. (If H is trivial Ω (G,H), is just a dot, whereas Γ (G, H) is complete).

Proof : Let O_1, O_2, ..., O_s be the complete set of distinct orbits of IrrH under the conjugate action of G an IrrH. Then by Clifford's theorem it is clear that two distinct θ, ϕ, $\in$ IrrH are adjacent in Ω (G,H) if and only if θ and ϕ lie in the same orbit O_i for some i. Since H Δ G, s >1 and hence Ω (G,H) is disconnected.

Infact, Ω (G,H) has exactly s components each of which is complete.

We shall push a little further in the compare and contrast syndrome vis-à-vis Γ (G, H) of Gnanaselan.

As per (1.3), Γ (G, (1)) is complete But in Ω (G,H) completeness can be achieved in quite different situations. First we shall recall Mackey's subgroup theorem.

3) *Theorem (Mackey [1]) :* If H is a subgroup of G and if θ, ϕ, $\in$ IrrH, then (θ^G, ϕ^G) = $\sum_{x \in D} (\theta, {}^x\phi)_{H \cap {}^xH}$, where D is a set of double coset representatives of H in G.

We now prove the following theorem providing a sufficient condition for the completeness of Ω (G,H)

4) *Theorem ;* Let H be a subgroup of G such that for some x $\in D$, the set of (H, H) - double coset representatives of H, H $\cap^x H$ = (1). Then Ω (G,H) is complete.

Proof : By Mackey's subgroup theorem $\theta^G, \phi^G = \sum_{x \in D} (\theta, {}^x\phi)_{H \cap {}^xH}$ for any two $\theta, \phi, \in$ IrrH,. By assumption, H $\cap^x H$ = (1) for some x $\in$ D. Hence $(\theta, {}^x\phi)_{(1)} > 0$, which implies $(\theta^G, \phi^G) > 0$. Since this is true for any two distinct $\theta, \phi, \in$ IrrH, it follows that Ω (G,H) is complete.

5) *Corollary :* If H is non-normal and prime cyclic, then Ω (G,H) is complete.

Proof is clear.

6) *Corollary :* If G = NH is a Frobenius group with complement H, then Ω (G,H) is complete.

Proof : By definition, H $\cap^a H$ = (1) for all a $\notin$ H . The result now follows from the above theorem.

Having made the above observations, it is perhaps worthwhile to point out the significance of both Γ(G, H) and Ω (G, H) both graph theoretically and group-theoretically. As was mentioned earlier, Gnanaseelan has highlighted many graph – theoretic properties of Γ (G, H). Further study of Γ(G,H) in graph – theoretic aspects such as colouring domination etc may throw more light on the study of character theory of G itself. It is interesting to note that the vertex sets of Γ(G,H) and the Brauer graph are one and the same and therefore comparisons are possible.

Turning to our Ω (G,H) we observe that this graph has a strong bearing on the character theory of algebraic groups and the related finite groups. The reasons are obvious. The study of the character theory of these groups heavily rests on the so-called 'Harish – Chandra Induction' and all the fascinating theory governing these aspects can be naturally fitted graph theoretically into our graph Ω (G,H). We propose to take up this study systematically in our future works, but for the moment, we shall briefly outline the construction of Ω (G,B), where G = GL (2,q) and B = the Borel subgroup, the subgroup of upper triangular matrices, ($q = p^n$ for a fixed prime p) and also G = PSL (2, q) and B = Borel subgroup. For a brief reference of the representation theory of G, we refer to [1]. We straightaway go to the following.

7) *Proposition ([1], p.22):* Let $\phi, \theta \in$, IrrB. Then $[\phi^G, \phi^G] = |w \in W \mid \phi^w = \phi|$ Where W is the weyl group of G and ϕ^w denotes the characters obtained by the conjugate action of w on ϕ.

8) *Proposition:* When G = GL (2, q), the graph Ω (G, B) is connected.

Proof : As already mentioned, W consists of just two elements namely, 1 and the element = $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (7) gives all the non-principal characters $\theta_{m,n}$ of Ω which can be pulled back to give all non-principal irreducible characters of B. It can be easily seen that $\theta^{\sigma}_{m,n} = \theta_{n,m}$. (Note that only when m $\neq$ n the pull backs are irreducible.)

Hence by (7)

$$(\theta^G_{m,n}, \theta^G_{n,m}) = |w \in W| \theta^w_{m,n} = \theta_{n,m}| = 1$$

Hence (T($\theta_{m,n}$), T ($\theta_{n,m}$) is non zero for all non - principal irreducible characters of B and therefore by the definition of the graph Ω(G, B) all the non-principal characters of B are adjacent. Since 1_B^G must have some irreducible other than 1_G we have Ω(G, B) is connected.

The case G = PSL (2, q), (q - p^n for a prime p), the Projective Special Linear Group of rank 1. The subgroup that we choose is as before the Borel subgroup B - TU, where T consists of the 2 x 2 matrices of determinant 1 and U denotes a p - Sylow subgroup (the unipotent subgroup). We have O (T) = $\frac{q-1}{2}$ when p is odd and q-1 when p is 2, O(U) = q and O (G) = $\frac{q(q^2-1)}{2}$ when q is odd and $q(q^2-1)$ when p is 2.

For the complete table of (Complex) irreducible representations of G, we refer to [2] (of course, we should pick out the irreducibles of G from this table, which is quite easy). Now B is a Frobenius group with kernel U and complement T.

We should distinguish two cases now, the p odd case and the p = 2 case.

I. The odd prime case.

Case 1. q $\equiv$-1 (mod 4)

In this case, these are $\frac{q+5}{2}$ distinct irreducible cases of G which are denoted as follows :

χ_1 (= the trivial character), χ_2 (= the Steinberg character of degree q), χ_3, χ_4, the irreducible characters of degree $\frac{q-1}{2}$, the $\frac{q-3}{4}$ irreducible characters $\psi_1, \psi_2, \ldots\psi_{\frac{q-3}{4}}$ each for degree q+1 and the $\frac{q-3}{4}$ irreducible characters $\psi'_1, \psi'_2, \ldots\psi'_{\frac{q-3}{4}}$ each of degree q-1.

From the character theory of Frobenius groups, we can easily see that B has exactly $\frac{q+3}{2}$ irreducible characters denoted by $\{\Phi_i\}_{1\le i\le\frac{q-1}{2}}$ (ϕ_1 = trivial character $\neq 1_B$), α_1 and α_2 each of degree $\frac{q-1}{2}$.

We can very easily calculate the character θ^G, for any $\theta \in \mathrm{Irr}B$. We write down the induced covers I (θ) as follows :

I (1_B) = $\{1_G, \chi_2\}$, (1)

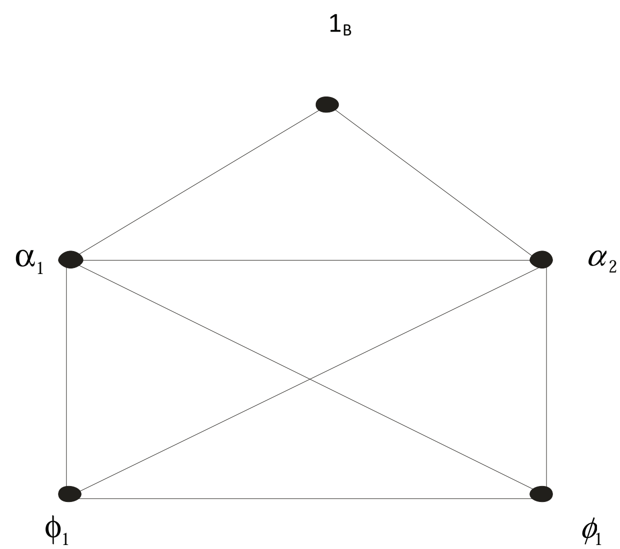
I (α_1) = $\{\chi_2, \chi_3, \{\psi_j\}_{1\le i\le\frac{q-3}{2}}, \{\psi'_k\}_{1\le k\le\frac{q-3}{2}}\}$ (2)

I (α_2) = $\{\chi_2, \chi_4, \{\psi_j\}_{1\le j\le\frac{q-3}{2}}, \{\psi'_k\}_{1\le k\le\frac{q-3}{2}}\}$ (3)

I(ϕ_i) = ψ_j (for some j) ($2 \le i \le \frac{q-1}{2}$) (4)

We have now proved the following theorem, whose proof can be gleaned from the equations (1), (2), (3) and (4).

9) Theorem : The graph Ω (G,H) is connected.We shall draw the graph Ω (G,B) for the case p = 7.



Case 2 : q $\equiv$ 1 (mod 4)

The $\frac{q+5}{2}$ irreducible characters of G are denoted by χ_1, χ_2 (= Steinberg), χ_3, χ_4, of degrees $\frac{q+1}{2}$, the $\frac{q-5}{4}$ irreducible characters

$\psi_1, \psi_2, \ldots, \psi_{\frac{q-5}{4}}$, each of degree q+1 and the $\frac{q-1}{4}$ characters $\psi'_1, \ldots \psi_{\frac{q-1}{4}}$ each of degree q-1.

The irreducible characters of B in this case also are 1_B, α_1, α_2 and $\{\phi_i\}_{1\le i\le \frac{q-1}{2}}$.

The induced covers in this case are given below.

$$I(1_B) = \{1_G, \chi_2\}, \quad (1)$$

$$I(\alpha_1) = \{\chi_2, \chi_3, \{\psi_j\}_{1\le i\le \frac{q-5}{4}}, \{\psi'_k\}_{1\le k\le \frac{q-1}{4}}\} \quad (2)$$

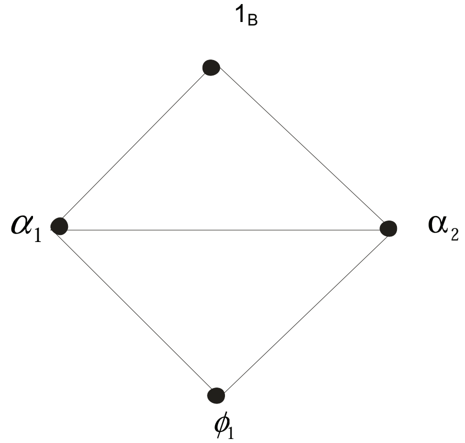
$$I(\alpha_2) = \{\chi_2, \chi_4, \{\psi_j\}_{1\le j\le \frac{q-5}{4}}, \{\psi'_k\}_{1\le k\le \frac{q-1}{4}}\} \quad (3)$$

$$I(\phi_i) = \psi_j \text{ (for some j) } (2 \le i \le \frac{q-1}{2}) \quad (4)$$

We have

10) Theorem : The graph Ω (G, B) is connected.

We shall draw the graph Ω (G, B) for the case when p = 5.



IV. THE GRAPHS Γ (G,H) AND Ω (G, H) FOR G – THE SYMMETRIC GROUP S_N.

We have already noted in (2) that even in the special case of groups given by the sequence H $\subset$ G $\subset$ L, Γ (G,H) and Ω (L,G) need not be isomorphic. However, it is interesting to note that in the case of the sequence of groups given by $S_{n-1} \subset S_n \subset S_{n+1}$, Γ (S_n, S_{n-1}) and Ω (S_{n+1}, S_n) are indeed isomorphic. The rest of the paper deals with a proof of this statement.

It is well known that the study of irreducible representations of S_n can be made through partitions of n.

1) Partitions of N :

A decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash$ i.e $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_r \ge 1$ is called a partition of n if $\lambda_1 + \lambda_2 + \ldots + \lambda_r$ = n. We denote this by the symbol $\lambda \vdash$ i.e λ n The λ_i's are called the parts of the partition λ, and the integer r is called the number of parts or length of λ.

Two partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ and = $\mu(\mu_1, \ldots, \mu_s)$ of n are said to be equal if r = s and $\lambda_1 = \mu_i$ for all i.

We shall omit several details regarding the well-known theory of representations via partitions as developed by Frobenius and Schur and just state the basic facts needed in the sequel.

2) The set of all conjugacy classes of S_n is naturally bijective with the set of all partitions of n. Since the set of all conjugacy classes of S_n is naturally bijective with the set of all complex irreducible characters of S_n, it immediately follows that the set of all complex irreducible characters of S_n is naturally bijective with the set of all partitions of n.

3) (Frobenius - Schur) : To every partition λ of n, one can associate an irreducible module $V\lambda$ for S_n. The family $(V\lambda) \vdash \lambda$ n is a complete set of mutually inequivalent irreducible representations of S_n over $\mathbb{C}$.

4) The Branching Theorem ([5]): If λ is a partition of n, we shall denote by $[\lambda]$ the unique irreducible representation $V\lambda$ associated to λ. We shall simultaneously denote the irreducible (complex) character associated to $V\lambda$ also by $[\lambda]$. Given a subgroup H of a finite group G, and given $\theta \in$ IrrH and $\chi \in$ IrrH respectively, we shall denote by $\theta\uparrow_H^G$ the induced character of θ to G and by $\chi\downarrow_H^G$ the restriction of χ to H.

5)Branching theorem for S_n.

Let $S_{n-1} \subseteq S_n \subseteq S_{n+1}$ in a natural way. Given $\lambda = (\lambda_1, \ldots, \lambda_r)$ n, let $\lambda^{j\pm}$ denote the partition $(\lambda_1, \ldots, \lambda_j \pm 1, \ldots \lambda_r)$ of n $\pm$ 1 according as $\lambda_{j-1} > \lambda_j$ or $\lambda_j > \lambda_{j+1}$.

Then we have

$$[\lambda]\downarrow_{S_{n-1}}^{S_n} = \bigoplus_{\substack{j \\ \lambda_j > \lambda_{j+1}}} [\lambda^{j-}] \text{ and}$$

$$[\lambda]\uparrow_{S_n}^{S_{n+1}} = \bigoplus_{\substack{j \\ \lambda_{j-1} > \lambda_j}} [\lambda^{j+}]$$

6) Lemma :

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be two partitions of n. Then $[\lambda]\uparrow_{S_n}^{S_{n+1}}$ and λ have a common irreducible constituent if and only if

i) $|r-s| \leq 1$,

ii) $|\lambda_i - \mu_i| \leq 1$ for every i (taking $\mu_{s+1} = 0$ when r = s+1 and $\lambda_{r+1} = 0$ when s = r +1 respectively.

iii) $|\lambda_i - \mu_i|$ = 1 for exactly two distincts i's

= 0 for the remaining i's.

Proof : Let $[\lambda]\uparrow$ and $[\mu]\uparrow$ have a common irreducible constituent. (Note that we have simplified the notation, since there is no danger of confusion).

(i) Suppose that |r-s| > 1.

Assume that r = s + k, k > 1. That is, λ has s + k parts and μ has s parts with k > 1. By Branching Theorem, the partition corresponding to any irreducible constituent of $[\lambda]\uparrow$ will have atleast s + k parts and the partition corresponding to any irreducible constituent of $[\mu]\uparrow$ will have atmost s+1 parts. But s + 1 < s + k as k > 1. Hence no irreducible constituent of $[\lambda]\uparrow$ equals any irreducible constituent of $[\mu]\uparrow$, which proves (i).

(ii) Assume that $|\lambda_i - \mu_i| > 1$ for some i, $1 \leq i \leq r$.

First let r = s. Suppose $\mu_i - \lambda_i > 1$.

Let $\lambda = (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_r)$ and

$\mu = (\mu_1, \ldots \lambda_i + k_i, \ldots, \mu_r)$, $k_i > 1$.

Then, in the partitions corresponding to each irreducible constituent of $[\lambda]\uparrow$, the i^{th} part is either λ_i or $\lambda_i + 1$ whereas the i^{th} part of the partitions corresponding to each irreducible constituent of $[\mu]\uparrow$ contains entries which are atleast $\lambda_i + k_i$. Hence no two irreducible constituents of $[\lambda]\uparrow$ and $[\mu]\uparrow$ will coincide.

Similar argument holds when $\lambda_i - \mu_i > 1$ for some i. When r = s + 1, $\lambda_r = 1$ and $\mu_r = 0$ and when s = r + 1, $\mu_s = 1$ and $\lambda_s = 0$, we can prove (ii) using similar arguments, which completes the proof of (ii).

(iii) Suppose that $|\lambda_i - \mu_i|$=1for k i's, k > 2. Since

$\Sigma\lambda_i = \Sigma\mu_i = n$, if $\lambda_i = \mu_i - 1$, then there exists some j such that $\lambda_j = \mu_j + 1$ to maintain parity. By Branching theorem, only one part of $[\lambda]$ and $[\mu]$ will be increased by at a time, which constitute the parts of $[\lambda]\uparrow$ and $[\mu]\uparrow$ respectively. Therefore no irreducible constituent of $[\lambda]\uparrow$ will coincide with any irreducible constituent of $[\mu]\uparrow$. That is, $[\lambda]\uparrow$ and $[\mu]\uparrow$ will not have a common constituent. Using similar arguments we can prove the result when $\mu_i = \lambda_i - 1$. Hence the result. Conversely, let the three conditions hold.

Case (i) : Let $|r - s| = 1$. It is enough to prove the result for $r = s+1$. Then $\lambda = (\lambda_1, \dots \lambda_{s+1})$ and $\mu = (\mu_1, \dots, \mu_s)$. We can take $\mu_{s+1} = 0$ and hence by (ii) $\lambda_{s+1} = 1$. By iii) there exists some j such that $\mu_j = \lambda_j + 1$ and for all other i, $\lambda_i = \mu_i$. That means,

$\mu = (\lambda_1, \dots \lambda_{i-1}, \lambda_i, \dots \lambda_s, 1)$ and $\mu = (\lambda_1, \dots \lambda_{j-1}, \lambda_{j+1}, \dots \lambda_s, 0)$. Now, $\lambda_{j-1} \geq \lambda_j + 1$. That is $\lambda_{j-1} > \lambda_j$. Therefore $(\lambda_1, \dots, \lambda_j+1, \dots \lambda_s$, is a constituent of $[\lambda]\uparrow$ clearly, replacing 0 by 1 in

1) $[\mu]\uparrow$ we get $(\lambda_1, \dots, \lambda_{j-1}, \lambda_j+1, \dots \lambda_s, 1)$
2) is a constituent of $[\mu]\uparrow$ as well.

Case (ii) :

$r = s$. Then $\lambda = (\lambda_1, \dots, \lambda_s)$ and $= (\mu_1, \dots, \mu_s)$. By (iii), there exists some j and k such that $\lambda_j = \mu_j + 1$ and $\mu_k = \lambda_k + 1$ and $\lambda_j = \mu_i$ for all other i's. That is, $\lambda = (\lambda_1, \dots, \lambda_{j-1}, \mu_j + 1, \dots, \lambda_k, \dots, \lambda_s)$ and $\mu = (\lambda_1, \dots, \lambda_{j-1}, \mu_j, \dots, \lambda_k + 1, \dots \lambda_s)$ Then $\lambda_{j-1} > \mu_j$. Therefore, $(\lambda_1, \dots, \lambda_{j-1}, \mu_j + 1, \dots, \lambda_k + 1, \dots \lambda_s)$ is a constituent of $[\mu]\uparrow$. Again $\lambda_{k-1} > \lambda_k$. Therefore $(\lambda_1, \dots \lambda_{j-1}, \mu_j + 1 \dots \lambda_{k-1}, \lambda_k + 1, \dots \lambda_s)$ is constituent of $[\lambda]\uparrow$ as well. This completes the proof of the lemma.

7) Lemma : Let $\lambda = (\lambda_1, \dots \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ be partitions of n. Then $[\lambda]\downarrow_{S_{n-1}}^{S_n}$ and $[\mu]\downarrow_{S_{n-1}}^{S_n}$ have a common irreducible constituent if and only if

i) $|r-s| \leq 1$,

ii) $|\lambda_i - \mu_i| \leq 1$ for every i (taking $\mu_{s+1} = 0$ when $r = s+1$ and $\lambda_{r+1} = 0$ when $s = r + 1$ respectively).

iii) $|\lambda_i - \mu_i| = 1$ for exactly two distinct i's

= 0 for the remaining i's.

Proof : Let $[\lambda]\downarrow$ and $[\mu]\downarrow$ have a common irreducible constituent (as before, we have simplified the notations).

(i) Suppose that $|r-s| > 1$.

Assume that $r = s+k$, $k>1$. By Branching theorem, the partition corresponding to any irreducible constituent of $[\lambda]\downarrow$ will have atleast s+k-1 parts and the partition corresponding to any irreducible constituent of $[\mu]\downarrow$ will have almost s parts. Since $k > 1$, $s + 1 < s + k$. Hence no irreducible constituent of $[\lambda]\downarrow$ is equal to any irreducible constituent of $[\mu]\downarrow$ which proves (i).

(ii) Assume that $|\lambda_i - \mu_i| > 1$ for some i, $1 \leq i \leq r$.

Let r = s Suppose $\mu_i - \lambda_i > 1$. Let $\lambda = (\lambda_1, \dots, \lambda_i, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots \mu_i + k_i, \dots, \mu_r)$, $k_i > 1$. Then in the partitions corresponding to each irreducible constituent of $[\lambda]\downarrow$, the i^{th} part is either λ_i or $\lambda_i - 1$ whereas the i^{th} part of the partitions corresponding to each irreducible constituent of $[\mu]\downarrow$ contain entries which are atleast $\lambda_i + k_i - 1$. But $k_i > 1$.

Hence no two irreducible constituents of $[\lambda]\downarrow$ and $[\mu]\downarrow$ will coincide.

Similar argument holds when $\lambda_i - \mu_i > 1$ for some 1. The cases r = s +1 and s = r + 1 can be dealt with similarly, completing the proof of ii).

(iii) Suppose that $|\lambda_i - \mu_i|=1$ for k i's, k > 2.

Since $\Sigma\lambda_i = \Sigma\mu_i = n$, if $\lambda_i = \mu_i + 1$, then there exists some j such that $\lambda_j = \mu_j -1$ to maintain parity. By Branching theorem, only one part of $[\lambda]$ and $[\mu]$ will be increased by 1 at a time, which constitute the parts of $[\lambda]\downarrow$ and $[\mu]\downarrow$ respectively. Hence if k > 2, there will be parts say λ_k, μ_k in $[\lambda]$, $[\mu]$ respectively ($k \neq 1,2$) such that $\lambda_k \neq \mu_k$ and these parts will not be touched upon restriction. Thus in each part of $[\lambda]\downarrow$ and $[\mu]\downarrow$, there will be some $\lambda_k \neq \mu_k$. That is, $[\lambda]\downarrow$ and $[\mu]\downarrow$ will not have any common constituent. Similarly we can prove this result when $\mu_i = \lambda_i-1$. Hence the result.

Conversely, let the three conditions hold.

Case (i) :

Let | r – s | =1. It suffices to prove the result for r = s+1. Then $\lambda = (\lambda_1, \ldots \lambda_{s+1})$ and $\mu = (\mu_1, \ldots, \mu_s)$. We can take $\mu_{s+1} = 1$ and hence by (ii) $\lambda_{s+1} = 0$. By iii) there exists some j such that $\mu_j = \lambda_j-1$ and for all other i, $\lambda_i = \mu_i$. That is, $\lambda = (\lambda_1, \ldots \lambda_{i-1}, \lambda_i, \ldots \lambda_s, 1)$ and $\mu = (\lambda_1, \ldots \lambda_{j-1}, \lambda_{j-1}, \ldots \lambda_s, 0)$. Now, $\lambda_{j-1} \geq \lambda_j - 1$. since otherwise in $\lambda = (\lambda_1, \ldots \lambda_{j-1}, \lambda_j, \ldots)$, contradiction. Hence $\lambda_{j-1} > \lambda_j$. Therefore, $(\lambda_1, \ldots, \lambda_{j-1}, \lambda_j, \ldots 0)$ is a constituent of $[\mu]\downarrow$ clearly the above is a constituent of $[\lambda]\downarrow$ as well.

Case (ii) :

Let r = s. Then $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\mu = (\mu_1, \ldots, \mu_s)$. By (iii), there exists some j and k such that $\lambda_j = \mu_j-1$ and $\mu_k = \lambda_k - 1$ and $\lambda_i = \mu_i$ for all other i's. That is, $\lambda = (\lambda_1, \ldots, \lambda_{j-1}, \mu_j-1, \ldots, \lambda_k, \ldots, \lambda_s)$ and $\mu = (\lambda_1, \ldots, \lambda_{j-1}, \mu_j, \ldots, \lambda_k-1, \ldots \lambda_s)$ Then $\lambda_{j-1} > \mu_j$. Therefore, $(\lambda_1, \ldots, \lambda_{j-1}-1, \mu_j \ldots, \lambda_k-1, \ldots \lambda_s)$ is a constituent of $[\mu]\downarrow$. Again $\lambda_{k-1} > \lambda_k$. Therefore $(\lambda_1, \ldots \lambda_{j-1}, \mu_j \ldots \lambda_{k-1} -1, \lambda_k, \ldots \lambda_s)$ is a constituent of $[\lambda]\downarrow$ also.

This completes the proof of the lemma. We are now in a position to prove our main theorem.

8)Theorem : The two graphs $\Omega(S_{n+1}, S_n)$ and $\Gamma(S_n, S_{n-1})$ are isomorphic.

*Proof :*Let $[\lambda]$ and $[\mu]$ belong to IrrS_n and let $[\lambda]\uparrow$ and $[\mu]\uparrow$ contain an irreducible character of S_{n+1} in common. Then by Lemma 5 and Lemma 6 $[\lambda]\uparrow$and $[\mu]\uparrow$ contain an irreducible character of S_{n-1} in common. Again by the two lemmas, the converse statement also holds. Hence $\Gamma(S_n, S_{n-1})$ and $\Omega(S_{n+1}, S_n)$ are isomorphic graphs.

V.CONCLUSION

The graphs Γ (G, H) and Ω (L,G) are generally different for $H\subset G\subset L$, as we observed earlier. Taking the cue from our result for S_n, it would be interesting to find conditions (in general) on H, G and L so that Γ (G,H) and Ω (L,G) are isomorphic.

Also a deep study of Ω(G,H) in the case of algebraic groups and suitable subgroups may throw more light on the representation theory of such groups.

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