LAG BASED HERMITE INTERPOLATION METHOD FOR SOLVING A ROOT OF NONLINEAR EQUATIONS

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Abstract- In this paper we propose a learning based iterative numerical method for computing a root ξ of a non-linear equation of the form $f(x) = 0$ in the interval $[a,b]$. Lag based method through Hermite interpolation modeled root discovery approach is developed and demonstrated in this paper. The new method has been tested for a series of functions considered by several researchers. The numerical experiments show that the new method is effective and converges faster to a root with lesser number of iterations and lesser number of function evaluations.

Keywords- Bisection; Secant; Regula falsi; Newton; Steffensen's; nonlinear equations; root finding; order of convergence; Hermite interpolation; iteration method

1. Introduction

Intelligent searching strategies will enhance the performance and efficiency of an iterative numerical method for solving a nonlinear equation $f(x)=0$.

If f is a continuous function on the interval $[a,b]$, the Bisection method converges to a root of f, which repeatedly bisects an interval, then selects a subinterval in which a root must lie for further processing. This is called a bracket of a root and this method is not constrained by any other parameter other than sign. So, this method is relatively slow.

Secant method uses a succession of roots of secant lines to better approximate a root of a function $f(x)$. The Secant method does not require that the root remain bracketed like the Bisection method, and hence it does not always converge but reduces computational time.

The classical Regula falsi method finds a simple root of the nonlinear equation $f(x)=0$ by repeated linear interpolation between the two current bracketing estimates. Regula falsi method combines features from the Bisection method and the Secant method, here sign as well as magnitude is used. If the initial end points a and b are chosen such that $f(a)$ and $f(b)$ are of opposite signs, then this assumption guarantees the existence of a zero of $f(x)$ in the interval [a,b]. This means that the Regula falsi method always converges. In Regula falsi method, at every iteration one point is dropped, the information available at points which are dropped recently can be used for approximating the function with better accuracy. This enhances the knowledge in the current support about the function.

Alefeld and Potra [1,2] proposed three efficient methods for enclosing a simple zero ξ of a continuous function $f(x)$ in the interval $[a,b]$, provided that $f(a)\, f(b) < 0$. Starting with the initial enclosing interval $[a_1,b_1] = [a,b]$, this method fits a quadratic polynomial to produce a sequence of intervals such that $\{(b_n - a_n)\}_{n=1}^{\infty}$

$$\xi \in [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \subseteq .... \subseteq [a_1, b_1] \subseteq [a,b].$$
$$\lim_{n\to\infty}(b_n - a_n) = 0$$

In a subsequent study by Frontini [6], a numerically comparable method through fitting a Hermite interpolating polynomial was suggested. Here in this method, by taking $x_0 = a$ and $x_1 = b$ as the starting points, with $f(x_0)f(x_1) < 0$, then x_{i+1} is taken as the root of Hermite polynomial $P_2[f, x_{i-1}, x_i](x)$ which belongs to (x_{i-1}, x_i). Use of the dichotomy condition $f(x_{i-1})f(x_i) < 0$ ensures the existence of at least one root of $P_2[f, x_{i-1}, x_i](x)$ belonging to (x_{i-1}, x_i) because of the continuity of $P_2[f, x_{i-1}, x_i](x)$. The author further suggested in a similar manner fitting a cubic Hermite polynomial $P_3[f, x_{i-1}, x_i](x)$ and a generalized method satisfying the interpolation conditions:

$$P_{2n-1}^{(k)}[f, a, b](a) = f^{(k)}(a)$$
$$P_{2n-1}^{(k)}[f, a, b](b) = f^{(k)}(b) \qquad k=1,2\ldots n-1 \rightarrow (1)$$

By considering x_{n+1} as the root ξ of Hermite interpolating polynomial $P_{2n-1}[f, x_{n-1}, x_n](x)$ belonging to (x_{i-1}, x_i), the order of convergence of the iterative method obtained in [6] is given as

$p = \dfrac{n + \sqrt{n^2 + 4n}}{2}$. Frontini [6] proposed that his method works effectively only for fitting a cubic Hermite polynomial.

Based on the earlier methods which emphasize as memory based learning [1,2] and with reinforced learning by making use of derivative through Hermite interpolation as lag based learning [6], in this paper we attempt to propose an iterative method as follows.

2. The new method

Hermite interpolation is an extension of basic polynomial interpolation that not only matches discrete information at a set of points, but also matches the slope (or rate of change) at those points.

We consider $x_0 = a$ and $x_1 = b$ as the starting points, with $f(x_0)f(x_1) < 0$, a point c is taken as $a<c<b$. A fifth degree Hermite polynomial is fit for these three distinct points, their function values and the values of their corresponding first derivatives.

Hermite Interpolating Polynomial: The Hermite interpolating polynomial interpolates not only the function $f(x)$, but also first order derivatives at a given set of data points $(x_i, f(x_i), f'(x_i))$, i=0,1,2,…n.

The Hermite interpolating polynomial is given by

$$H_{2n+1}(x) = \sum_{i=0}^{n}\left[1 - 2(x - x_i)l_i'(x_i)\right]\left[l_i(x)\right]^2 f(x_i)$$
$$+ \sum_{i=0}^{n}(x - x_i)[l_i(x)]^2 f'(x_i) \qquad \rightarrow (2)$$

Where

$$l_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

We now consider fifth degree Hermite interpolating polynomial passing through the points $(x_i, f(x_i), f'(x_i))$, i=0,1,2 by putting n=2 in (2).

Subroutine

$f\_hermite(x_i, f(x_i), f'(x_i))$

$x_i = [a, c, b]$

$f(x_i) = [f(a), f(c), f(b)]$

$f'(x_i) = [f'(a), f'(c), f'(b)]$

x_r =root of $H_5(x)$

Algorithm

1. Set a=lower bound for search, b=upper bound for search such that $f(a)f(b) < 0$,find a point c such that $a < c < b$. Obtain corresponding $f'(a), f'(b), f'(c)$.

2. Call $f\_hermite(x_i, f(x_i), f'(x_i))$ which returns five roots of fifth degree Hermite polynomial.

3. From the above five roots select the root (say x_r) which lies in the interval [a,b] and for which function value $f(x_r)$ is minimum.

4. Obtain $f(x_r)$ if $f(x_r) < \in$, then display root= x_r

5. Compute $f'(x_r)$

6. If $f(a)\, f(x_r) < 0$, then

$$c = b,\ f(c) = f(b),\ f'(c) = f'(b)$$
$$\bar{a} = a, \bar{b} = x_r,\ f(\bar{b}) = f(x_r),\ f'(\bar{b}) = f'(x_r)$$

If $f(b)\, f(x_r) < 0$, then

$$c = a,\ f(c) = f(a),\ f'(c) = f'(a)$$
$$\bar{a} = x_r, \bar{b} = b,\ f(\bar{a}) = f(x_r),\ f'(\bar{a}) = f'(x_r)$$

Repeat step 2 until $| f(x) | < \in$.

In each iteration, the information about the point to be dropped is stored and used as lag, (always third point in new method is lag) to fit fifth degree Hermite interpolating polynomial, hence we need only two function evaluations at each iteration which improves the efficiency of the method.

3. **The order of convergence**

Let the function $f(x)$ defined by (n+1) points $(x_i, f(x_i))$, i=0,1,2,…n, be continuous and differentiable (n+1) times and let $f(x)$ be approximated by a polynomial $\phi_n(x)$ of degree not exceeding n such that

$\phi_n(x_i) = f(x_i)$, i=0,1,2...n, then error is given by

$$E(x) = f(x) - \phi_n(x)$$
$$= \pi_{n+1}(x)\, f[x_0, x_1, ..x_n] = \pi_{n+1}(x) L$$

where $\pi_{n+1}(x) = (x - x_0)(x - x_1)...(x - x_n)$

Then by constructing a function $F(x)$ such that $F(x) = f(x) - \phi_n(x) - L\pi_{n+1}(x)$ and the constant L can be determined so that

$$L = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

Hence we get

$$E(x) = f(x) - \phi_n(x)$$
$$= \frac{1}{(n+1)!} \pi_{n+1}(x)\, f^{(n+1)}(\xi),\ x_0 < \xi < x_n$$

Now, we consider Hermite interpolating polynomial $H_{2n+1}(x)$, then error $E(x) = f(x) - H_{2n+1}(x)$ can be obtained. Here we notice that both $E(x)$ and $[\pi_{n+1}(x)]^2$ vanish together with the first derivatives at each of (n+1) points. We then form a linear combination of these functions

$$F(x) = f(x) - H_{2n+1}(x) - L[\pi_{n+1}(x)]^2$$

$$E(x) = f(x) - H_{2n+1}(x) = \frac{[\pi_{n+1}(x)]^2}{(2n+2)!} f^{(2n+2)}(\xi) \rightarrow (3)$$

Where $f(x)$ is assumed to have continuous derivatives of order (2n+2) and $\xi = \xi(x)$ is in the interval determined by the points $x, x_0, \ldots, x_n$.

Theorem: Let $f \in C^6$, at least in a neighborhood of α, with f (α) = 0. Let $\{x_n\}$ be the sequence defined as $x_{n+1} = \xi$, where ξ is the root of fifth degree Hermite polynomial enhanced by the dichotomy procedure so that it converges to α with n. Let $\in_n = x_n - \alpha$ be the error of the nth iterate. Then, $\in_{n+1}| \sim M |\in_n|^{2..9196}$ where M is a positive constant.

Proof Without loss of generality we assume that $x_{n-2} < x_{n-1} < x_n$.

There exists $\eta \in]x_{n-2}, x_n[$ such that

$$E(\alpha) = f(\alpha) - H_5[f, x_{n-2}, x_{n-1}, x_n](\alpha) = \frac{(\alpha - x_{n-2})^2 (\alpha - x_{n-1})^2 (\alpha - x_n)^2}{6!} f^{(6)}(\eta) \rightarrow (4)$$

If x_{n+1} is zero of $H_5[f, x_{n-2}, x_{n-1}, x_n](x_{n+1})$ in $]x_{n-2}, x_n[$, then

$$H_5[f, x_{n-2}, x_{n-1}, x_n](x_{n+1}) = 0 = f(\alpha)$$

By replacing $f(\alpha)$ with $H_5[f, x_{n-2}, x_{n-1}, x_n](x_{n+1})$ in (4), we get

$$H_5[f, x_{n-2}, x_{n-1}, x_n](x_{n+1}) - H_5[f, x_{n-2}, x_{n-1}, x_n](\alpha) = \frac{(\alpha - x_{n-2})^2 (\alpha - x_{n-1})^2 (\alpha - x_n)^2}{6!} f^{(6)}(\eta)$$

By Lagrange's theorem, there exists $\xi \in]\alpha, x_{n+1}[$ such that

$$(\alpha - x_{n+1}) H_5'[f, x_{n-2}, x_{n-1}, x_n](\xi) = H_5[f, x_{n-2}, x_{n-1}, x_n](x_{n+1}) - H_5[f, x_{n-2}, x_{n-1}, x_n](\alpha) \rightarrow (5)$$

By comparing (4) and (5) we get $(\alpha - x_{n+1}) = K(\alpha - x_{n-2})^2 (\alpha - x_{n-1})^2 (\alpha - x_n)^2$

Where $K = \dfrac{f^{(6)}(\eta)}{6!\, H_5'[f, x_{n-2}, x_{n-1}, x_n](\xi)}$, $\xi \in]\alpha, x_{n+1}[$, $\eta \in]x_{n-2}, x_n[$

Assuming $\in_{n+1}$ is asymptotic to $c \in_n^p,\ p > 1$,

Expressing this in terms of $\in_{n-1}$, we get

$$(\alpha - x_{n+1}) = K(\alpha - x_{n-2})^2 (\alpha - x_{n-1})^2 (\alpha - x_n)^2$$

$$\in_{n+1} = K \in_{n-2}^2 \in_{n-1}^2 \in_n^2$$

$$c^{p+1} \in_{n-1}^{p^2} = K \in_{n-1}^{\frac{2}{p}} \in_{n-1}^2 \in_{n-1}^{2p}$$

This asymptotic equation is satisfied if p is the positive root of

$$\Rightarrow \quad p^2 = \frac{2}{p} + 2 + 2p$$

$$\Rightarrow \quad p^3 - 2p^2 - 2p - 2 = 0$$

$$\Rightarrow \quad p = 2.9196$$

then, $\in_{n+1} | \sim M | \in_n|^{2..9196}$ for a suitable constant *M*.

It is proved in the above theorem that the new method converges to a root of *f*(*x*) with rate of convergence 2.9196. Given three distinct points we obtain their function values and the values of first derivatives corresponding to these three points before starting the first iteration. Consecutively we need only two function evaluations at each iteration as lag based intelligent searching strategy is used which improves the Efficiency Index, given by $EI = (2.9196)^{\frac{1}{2}} = 1.708684$ .This *EI* is better than secant method, Newton method, method suggested by Costabile [3,4] and method proposed by Frontini [6].

Table I: Comparison of EI of various methods is shown in Table I.

Method	*n*	*p*	EI = $p^{1/n}$
Bisection	1	1	1
Secant	1	1.618	1.618
Regula Falsi	1	1	1
Newton	2	2	1.41
Frontini	2	1+ $\sqrt{3}$	1.6529
New method	2	2.9196	1.7087

4. NUMERICAL EXPERIMENTS

The results of numerical experiments are shown in the tables II, III and IV.

TABLES II and III :

The computed results of Examples 1-4 by new method with tolerance= 1×10^{-15} are given in Table II. Here NI stands for number of iterations and NFE stands for number of function evaluations. Table III- displays results obtained by Exponential regulafalsi, Regulafalsi, Steffensen's and Newton methods(see[7])

TABLE II

s.no	Function f (x)	Interval for Search	Root ξ	$\lvert f(\xi)\rvert$	NI	NFE
1	lnx	[0.5,5]	1.00000000000000	2.4424907e-016	4	13
2	$x+1-e^{\sin x}$	[1, 4]	1.69681238680976	2.6541269e-016	3	11
3	$11x^{11}-1$	[0.1,1]	0.80413309750367	6.5052130e-016	4	13
4	$xe^{-x}-0.1$	[0,1]	0.11183255915896	4.9786564e-016	2	9

	Algorithm 2 (EXRF)			Algorithm 1 (regula falsi)			Steffensen	Newton	
	NI	ξ	$\lvert f(\xi)\rvert$	*NI*	ξ	$\lvert f(\xi)\rvert$	*NI* ξ $\lvert f(\xi)\rvert$	*NI* ξ	$\lvert f(\xi)\rvert$
1	7	1.00000e+00	0.00000e+00	27	1.00000e+00	8.88178e-16	Failure	Divergent	
2	11	1.6968.e+00	4.44089e-16	32	1.6968e+00	4.44089e-16	Failure	Not convergent	
3	8	8.04133e-01	4.44089e-16	101	8.04133e-01	1.25422e-13	Divergent	7 8.04133e-01	4.44089e-16
4	6	1.11833e-01	0.00000e+00	15	1.11833e-01	7.4040le-16	Failure	Failure	

TABLE III (see [7]) *

*calculations given by Jinhai Chen, and Weiguo Li [7]

TABLE IV: In this Table IV we present results through new method for some examples considered by several researchers with tolerance= 1×10^{-15}

s.no	Function f (x)	Interval for Search	Root	$\|f(\xi)\|$	NI	NFE
1	$x^2(x^2/3+\sqrt{2}\sin x)-\sqrt{3}/18$	[0.1,1]	0.39942229171099	3.1918912e-016	3	11
2	$2xe^{-n}+1-2e^{-nx}$ n=5,10,20	[0,1]	0.13825715505682 0.06931408868702 0.03465735902085	4.4408921e-016 6.9388939e-017 3.3306691e-016	3 4 4	11 13 13
3	$x^2-(1-x)^n$ n=5,10,20	[0,1]	0.34595481584824 0.24512233375331 0.16492095727644	4.3021142e-016 9.6450625e-016 9.3675068e-017	1 4 6	7 13 17
4	$e^{-nx}(x-1)+x^n$ n=5,10,20	[0,1]	0.51615351875793 0.53952222690850 0.55270466667922	3.4000580e-016 5.5250943e-016 1.5252099e-016	3 4 12	11 13 29
5	$x^2+\sin(x/n)-1/4$ n=5,10,20	[0,1]	0.40999201798914 0.45250914557763 0.47562684859606	2.7755576e-017 5.5511151e-017 4.9960036e-016	2 2 1	9 9 7
6	$(nx-1)/((n-1)x)$ n=5,10,20	[0.01,1]	0.20000000000000 0.10000000000000 0.05000000000000	4.7184479e-016 2.4980018e-016 2.1035805e-016	9 8 7	23 21 19
7	x^3-e^{-x}	[0,1]	0.77288295914921	4.9960036e-016	2	9
8	xe^x-1	[0,1]	0.56714329040978	6.6613381e-016	5	15
9	10^x+x-4	[0,1]	0.53917912205280	0.0000000e+000	3	11

5. CONCLUSION

As learning based intelligent searching technique, we propose a method of fitting a fifth degree Hermite polynomial (for three distinct points, their respective function values and values of their corresponding first derivatives). The present method, using lag based searching technique with tolerance= 1×10^{-15} has a rate of convergence 2.9196 and better Efficiency Index of 1.708684. The EI obtained by the present method is better than the EI of Secant method, Newton method, as well as the method suggested by Costabile [3,4] and the method proposed by Frontini [6]. Numerical experiments presented in the paper show that this method gives better performance and it is fast approaching to a root of given non linear equation $f(x) = 0$ with lesser number of iterations and lesser function evaluations and comparable to well known methods.

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